NON-ORIENTABLE FUNDAMENTAL SURFACES IN LENS SPACES

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ABSTRACT. We consider non-orientable closed surfaces of minimum crosscap number in the (p, q)-lens space $L(p, q) \cong V_1 \cup_\partial V_2$, where V_1 and V_2 are solid tori. Bredon and Wood gave a formula for calculating the minimum crosscap number. Rubinstein showed that $L(p, q)$ with p even has only one isotopy class of such surfaces, and it is represented by a surface in a standard form, which is constructed from a meridian disk in V_1 by performing a finite number of band sum operations in V_1 and capping off the resulting boundary circle by a meridian disk of V_2. We show that the standard form corresponds to an edge-path λ in a certain tree graph in the closure of the hyperbolic upper half plane. Let $0 = p_0/q_0,\ p_1/q_1,\ \cdots,\ p_k/q_k = p/q$ be the labels of vertices which λ passes. Then the slope of the boundary circle of the surface right after the i-th band sum is (p_i, q_i). The number of edges of λ is equal to the minimum crosscap number. We give an easy way of calculating p_i/q_i using a certain continued fraction expansion of p/q.

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1. INTRODUCTION

A solid torus V is homeomorphic to $D^2 \times S^1$, where D^2 is the 2-dimensional disk and S^1 the 1-dimensional sphere. A circle on the boundary torus $\partial V \cong (\partial D^2) \times S^1$ is of (p, q)-*slope* (or p/q-slope) if it is isotopic to the circle given by the expression $((\cos 2\pi q\theta, \sin 2\pi q\theta), (\cos 2\pi p\theta, \sin 2\pi p\theta))$ where p and q are coprime integers and θ is a parameter with $0 \leq \theta \leq 1$. We call a circle of $(1, 0)$-slope a *longitude* and that of $(0, 1)$-slope a *meridian*. In general, a circle in a surface is said to be *essential* if it does not bound a disk in the surface. As is well-known, in the boundary torus ∂V, any essential circle is of (p, q)-slope for some coprime integers p, q.

For a pair of coprime integers p and q with $p \geq 2$, the (p, q)-*lens space* $L(p, q)$ is obtained from two solid tori V_1 and V_2 by gluing their boundary tori by a homeomorphism which maps the meridian circle on ∂V_2 to a circle of (p, q)-slope on ∂V_1. Throughout this paper, we regard $L(p, q)$ as the union of V_1 and V_2 as above. Since $L(p, q) \cong L(p, -q)$ and $L(p, q) \cong L(p, q + p)$, it is enough for establishing a general result for (p, q)-lens spaces to consider $L(p, q)$ with $1 \leq q \leq 2/p$.

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It is well-known that $L(p,q)$ contains a non-orientable closed surface if and only if p is even. This holds because $H_2(L(p,q);\mathbb{Z}_2) = \mathbb{Z}_2$ when p is even, $H_2(L(p,q);\mathbb{Z}_2) = 0$ when p is odd, and $H_2(L(p,q);\mathbb{Z}) = 0$ for any integer $p \geq 2$. In [1], Bredon and Wood gave a formula for calculating the minimum crosscap number $\mathrm{Cr}(p,q)$ among those of all non-orientable connected closed surfaces in $L(p,q)$. The formula is based on a continued fraction expansion of p/q.

Notation 1.1. For a finite sequence of real numbers $a_0, a_1, \cdots, a_n$, we let $[a_0, a_1, \cdots, a_{n-1}, a_n]$ denote $a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}} \in \mathbb{R} \cup \{\infty\}$, where $r/0 = \infty$, $r/\infty = 0$ and $\infty + r = \infty$ for any real number r,

Definition 1.2. Let r be a positive rational number. A continued fraction expansion $r = [a_0, a_1, \cdots, a_n]$ is said to be *standard* if a_0 is a non-negative integer, a_i is a natural number for $i = 1, 2, \cdots, n$ and $a_n \geq 2$. By considering Euclidean method of mutual division, it is easily seen that such an expression is unique for r.

Theorem 1.3. (Bredon and Wood, (6.1) Theorem in [1]) *Let p, q be coprime natural numbers with $p \geq 2$, and $p/q = [a_0, a_1, \cdots, a_n]$ the standard continued fraction expansion.*

Then the minimum crosscap number is calculated by $\mathrm{Cr}(p,q) = \sum_{i=0}^{n} b_i/2$,
where $b_0 = a_0$, *and* $b_i = \begin{cases} a_i & (\text{when } b_{i-1} \neq a_{i-1} \text{ or } \sum_{j=0}^{i-1} b_j \text{ is odd}) \\ 0 & (\text{otherwise}) \end{cases}$

J. H. Rubinstein studied non-orientable closed surfaces in 3-manifolds in [5], [6] and [7]. Such surfaces of minimum crosscap number in lens spaces are considered in section 3 in [7]. See also [2], in which one-sided closed surfaces in Seifert fibered spaces are studied by C. Frohman. In order to introduce results in [7], we need to recall some definitions.

Definition 1.4. Let M be a closed 3-manifold, and F a (possibly one-sided) closed surface embedded in M. An embedded disk D in M is called a compressing disk of F if $D \cap F = \partial D$ and the boundary circle ∂D is essential in F. We say F is *geometrically incompressible* if it has no compressing disk. If it has, it is *geometrically compressible*.

Remark 1.5. A non-orientable closed surface of minimum crosscap number in $L(p,q)$ is geometrically incompressible as shown in lines 9–11 in page 192 in [7]. Section 2 includes the argument for self-containedness.

Definition 1.6. Let M be a compact 3-manifold with non-empty boundary ∂M, F a compact surface properly embedded in M, and β an arc embedded in ∂M so that $\beta \cap \partial F = \partial \beta$. We

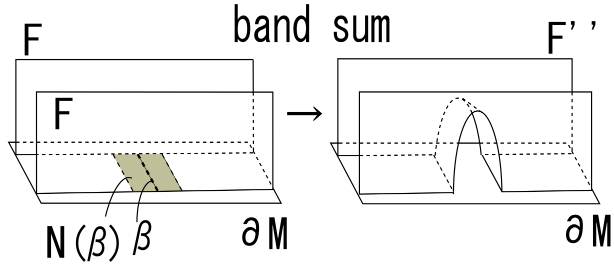

FIGURE 1.

obtain a new surface F'' by an operation called a *band sum* on F along β as below. We take a tubular neighbourhood $N(\beta) \cong \beta \times I$ of β in ∂M so that $\partial F \cap N(\beta) = (\partial\beta) \times I$. We isotope the interior of the surface $F' = F \cup N(\beta)$ slightly into $\operatorname{int} M$ with its boundary circles $\partial F'$ fixed, to obtain a new surface F''. See Figure 1. A band sum is *trivial* if a union of β and a subarc of ∂F forms a circle bounding a disk in ∂M.

Remark 1.7. In the above definition of band sum, if ∂M is a torus and ∂F is a single essential circle in ∂M, then a non-tirivial band sum is along an arc connecting the both sides of ∂F, that is, for an arbitrary tubular neighborhood $N(\partial F) \cong \partial F \times I$ of ∂F in ∂M, both $\beta \cap (\partial F \times \{0\}) \neq \emptyset$ and $\beta \cap (\partial F \times \{1\}) \neq \emptyset$ hold. In this case, the boundary of the resulting surface is again a single essential circle in ∂M.

Definition 1.8. Let F be a connected closed surface embedded in $L(p,q) \cong V_1 \cup_\partial V_2$. Then we say that F is in *standard form* if F is obtained from a meridian disk D_1 of V_1 by performing a finite number of non-trivial band sum operations and capping off the boundary circle of the resulting surface by a meridian disk D_2 of V_2.

Note that F in standard form is non-orientable, since it intersects a core loop of V_1 (resp. V_2) in a single point in the interior of D_1 (resp. D_2).

Theorem 1.9. (Rubinstein, Proposition 11 and Theorem 12 in [7]) *Let $L(p,q) \cong V_1 \cup_\partial V_2$ be the (p,q)-lens space with p even. Let F be a geometrically incompressible connected closed surface in $L(p,q)$ which is not homeomorphic to the 2-sphere. Then F can be isotoped to be in standard form. Moreover, in $L(p,q)$, a geometrically incompressible non-spherical connected closed surface is unique up to isotopy, and hence is a non-orientable closed surface of minimum crosscap number.*

As an example, the sequence of band sums for $(8,3)$-lens space is described in Figure 2.

Applications of uniqueness of an isotopy class of some kind of non-orientable surfaces are found in [5], [6], [7], [8] and [4].

The next theorem gives a new formula of the minimal crosscap number of non-orientable connected closed surfaces in $L(p,q)$, which is obtained from consideration on standard positions.

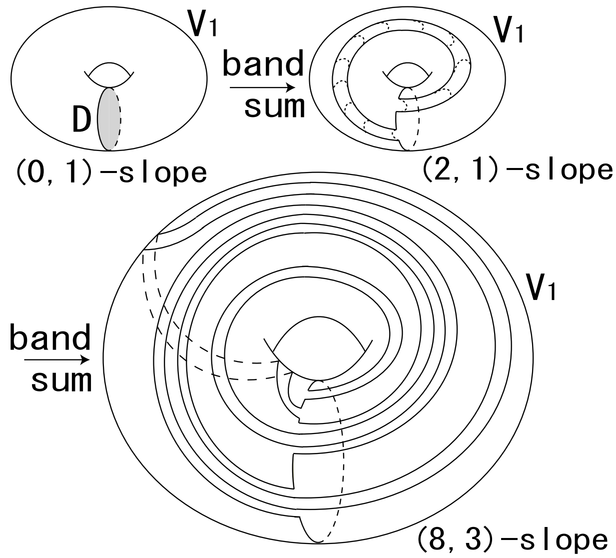

FIGURE 2.

Theorem 1.10. *Let p, q be coprime natural numbers with $p \geq 2$, and $p/q = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_0]$ be the standard continued fraction expansion.*

We define $\alpha'_0 = \alpha_0$, *and* $\alpha'_i = \begin{cases} \alpha_i \text{ (when } \alpha'_{i-1} = \infty) \\ \alpha_i + 1 \text{ (when } \alpha'_{i-1} \text{ is odd)} \\ \infty \text{ (when } \alpha'_{i-1} \text{ is even)} \end{cases}$ *for* $i = 1, 2, \cdots, n$.

We set $\beta_i = \begin{cases} \alpha'_i/2 \text{ (when } \alpha'_i \text{ is even)} \\ (\alpha'_i - 1)/2 \text{ (when } \alpha'_i \text{ is odd)} \\ 0 \text{ (when } \alpha'_i = \infty) \end{cases}$ *for* $i = 0, 1, \cdots, n$.

Then the minimum crosscap number is calculated by $\mathrm{Cr}(p, q) = \sum_{i=0}^{n} \beta_i$.

In [3], transitions of slopes (of circles and arcs in a 2-sphere with four punctures) caused by band sum operations is described by edge-paths in the graph $\mathbb{D}$ below. In this paper, we introduce a certain tree graph $\mathbb{D}_2$ taking after $\mathbb{D}$.

For a pair of integers p and q, we say p/q is an *irreducible fractional number* if p and q are coprime, that is, $\mathrm{GCD}(p, q) = 1$. Hence $p = p/1$ is an irreducible fractional number for any integer p. We consider $1/0$ and $(-1)/0$ representing the same irreducible fractional number, which is denoted by ∞. As usual, $\infty + p/q = \infty$ and $1/\infty = 0$. For an arbitrary pair of irreducible fractional numbers p/q and r/s, we set $d(p/q, r/s) = |\det \begin{pmatrix} p & r \\ q & s \end{pmatrix}| = |ps - rq|$, and call it the *distance* of them.

The graph $\mathbb{D}$ is embedded on the upper half plane $\mathbb{H}$ ($\subset \mathbb{C}$) with the real line $\mathbb{R}$ and the point at infinity ∞. Its vertices are the rational points and ∞ in $\mathbb{R} \cup \{\infty\}$, and its edges are geodesics on the upper half model of the hyperbolic plane which connect two vertices corresponding to the irreducible fractional numbers a/c, b/d, $(a, b, c, d \in \mathbb{Z})$ if and only if

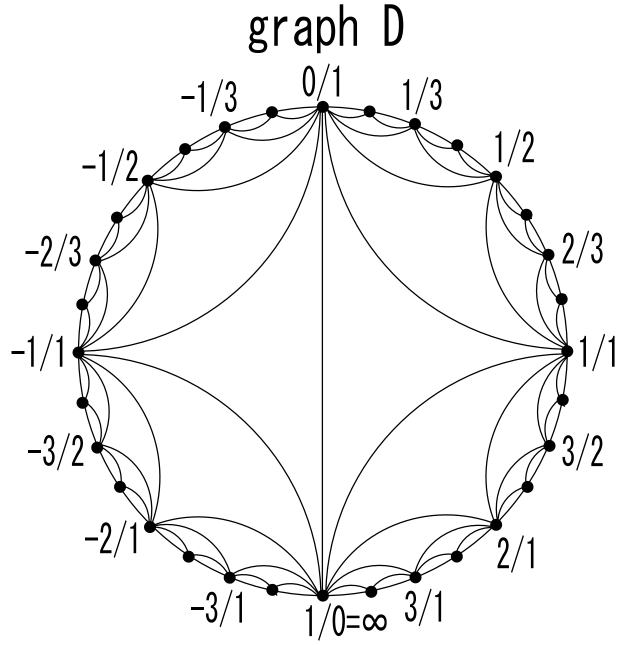

Figure 3.

$|ad - bc| = d(a/c, b/d) = 1$. See Figure 3, where $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ is transformed onto the Poincaré disk model by the map $z \mapsto \dfrac{z+i}{iz+1}$. We can easily draw this graph by following the rule that two vertices corresponding to the irreducible fractional number p/q, r/s and connected by an edge of $\mathbb{D}$ are those of a triangle face of $\mathbb{D}$ with $(p+q)/(r+s)$ being the third vertex.

When p is even, a vertex of $\mathbb{D}$ corresponding to the irreducible fractional number p/q is called an *even* vertex in this paper. If p is odd, we call it an *odd* vertex. Adequately separating the trigonal faces of $\mathbb{D}$ into adjacent pairs and taking a union of every pair of trigonal faces, we obtain a tiling of the upper half plane by infinitely many quadrilaterals with two even vertices and two odd vertices. The vertices of $\mathbb{D}_2$ are the even vertices of $\mathbb{D}$. Two vertices p/q, r/s, ($p, q, r, s \in \mathbb{Z}$, $\mathrm{GCD}(p, q) = 1$ and $\mathrm{GCD}(r, s) = 1$) are connected by an edge of $\mathbb{D}_2$ if and only if $|ps - rq| = d(p/q, r/s) = 2$. Each edge of $\mathbb{D}_2$ does not appear in $\mathbb{D}$, but forms a diagonal line of a quadrilateral of the tiling as above. See Figure 4, where $\mathbb{D}_2$ is described by solid lines. We regard the vertex, assigned an irreducible fractional number r/s, as corresponding to the (r, s)-slope on ∂V_1.

The next theorem will be shown in Section 4.

Theorem 1.11. *For any band sum operation in Definition 1.8, the slopes of the boundary circles of the surfaces in V_1 before and after the operation are connected by an edge of $\mathbb{D}_2$.*

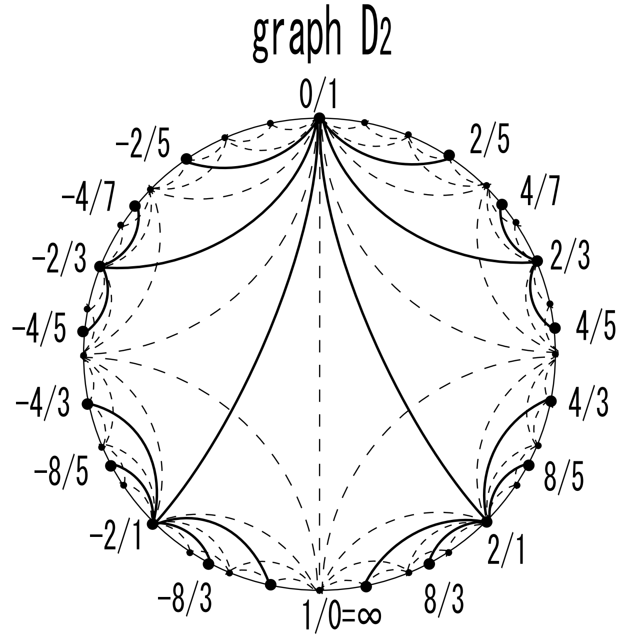

FIGURE 4.

This theorem implies that the transition of the slopes of the boundaries of the surfaces in V_1 by the band sums in Definition 1.8 is along an edge-path of $\mathbb{D}_2$, in which the same edge can appear twice.

We will show the next theorem in Section 5.

Theorem 1.12. *The graph $\mathbb{D}_2$ is a tree, i.e., $\mathbb{D}_2$ is connected and contains no cycle.*

This theorem also implies the uniquness in Theorem 1.9, as we give a proof in Section 6. However, Rubinstein's proof (of Theorem 12 in [7]) is very short and clear.

Theorem 1.13. *Let $L(p,q)$, V_1, V_2, F, F' be as in Theorem 1.9. F' is in standard form, and let D_1, D_2 be as in Definition 1.8. Set $D_1 = F_0$, and let F_i be the surface obtained from F_{i-1} by the i-th band sum operation in Definition 1.8. The tree $\mathbb{D}_2$ contains a unique minimal edge-path $\gamma(p,q)$ connecting $0/1$ to p/q, in which every edge appears at most once. Let $0/1 = r_0, r_1, \cdots, r_k = p/q$ be the vertices which $\gamma(p,q)$ passes in this order. Then the slope of the boundary circle ∂F_i is r_i, and $F' = F_k \cup D_2$. Moreover, the minimum crosscap number is equal to k, the number of the band sum operations. Suppose $q > 0$. Let $r_i = [a_0, a_1, \cdots, a_n]$ be the standard continued fraction expansion. Then $r_{i-1} = [a_0, a_1, \cdots, a_n - 2]$ for any integer i with $1 \leq i \leq k$.*

Section 2 contains the proof of Remark 1.5, and Section 3 that of the former half of Theorem 1.9 for self-containedness. We prove Theorem 1.11 in Section 4. In Section 5, Theorem 1.12 is shown. Theorems 1.10 and 1.13 are proved in Section 6.

2. Geometrical incompressibility

Lemma 2.1. *Let M be a compact, connected 3-manifolds, and F a non-separating closed surface of minimum crosscap number in M. Then F is geometrically incompressible.*

Proof. We assume, for a contradiction, that F is geometrically compressible. Let D be a compressing disk of F. We take a tubular neighborhood $N(D) \cong D \times I$ so that $N(D) \cap F = (\partial D) \times I$. We perform a surgery on F along D to obtain a new surface F', that is, we set $F' = (F - (\partial D) \times I) \cup (D \times \partial I)$.

First, we consider the case where ∂D is separating in F. Then F' consists of two connected components, say, F_1 and F_2, and $\chi(F) + 2 = \chi(F_1) + \chi(F_2) \cdots$ (i), where χ denotes the Euler characteristic. We assume, for a contradiction, that both F_1 and F_2 are separating in M. Then F_1 separates M into two components M_{1+} and M_{1-}, and F_2 into M_{2+} and M_{2-}. Without loss of generality, we can assume that $F_j \subset M_{i+}$ for $\{i, j\} = \{1, 2\}$. Then $N(D) \subset M_{1+} \cap M_{2+}$, and F separates M into $M_{1-} \cup N(D) \cup M_{2-}$ and $(M_{1+} \cap M_{2+}) - N(D)$. This contradicts that F is non-separating in M. Hence either F_1 or F_2, say, F_1 is non-separating. Because D is a compressing disk, F_2 is not a sphere and $\chi(F_2) \leq 1$, Hence the equation (i) implies $\chi(F) < \chi(F_1)$. This contradicts the minimality of $\chi(F)$.

Next, we consider the case where ∂D is non-separating in F. Let F_3 be the surface resulting from the surgery along D. We assume, for a contradiction, that F_3 separates M into two connected components, say, M_{3+} and M_{3-}. Without loss of generality, we assume $D \subset M_{3-}$ after the surgery. Then F separates M into $M_{3+} \cup N(D)$ and $M_{3-} - N(D)$. This contradicts that F is non-separating. Thus F_3 is non-separating in M. Since $\chi(F) + 2 = \chi(F_3)$, we obtain $\chi(F) < \chi(F_3)$. This contradicts the minimality of $\chi(F)$ again. □

Lemma 2.2. *In a lens space, a non-orientable closed surface with the minimum crosscap number is geometrically incompressible.*

Note that a similar thing does not hold for $S^2 \times S^1$.

Proof. Since $L(p, q)$ is an orientable 3-manifold, and since the 2-dimensional homology $H_2(L(p, q); \mathbb{Z}) = 0$, a closed surface F in $L(p, q)$ is non-orientable if and only if F is non-separating. Hence non-orientable closed surface of the minimum crosscap number is geometrically incompressible in $L(p, q)$ by Lemma 2.1. □

3. Standard form

In this section, the next propostion due to Rubinstein, restated using the terminology "band sum", is shown for self-containedness. The proof is almost the same as the original one in [7].

Proposition 3.1. (Rubinstein, Proposition 11 in [7]) *In $L(p,q) \cong V_1 \cup_\partial V_2$, any geometrically incompressible connected closed surface not homeomorphic to the 2-sphere is isotopic to a surface in standard form.*

Lemma 3.2. *Let F be a geometrically incompressible, connected closed surface in $L(p,q)$. We assume that $F \cap V_1$ is a disjoint union of meridian disks of V_1. If the number of meridian disks of $F \cap V_1$ is minimum up to isotopy of F in $L(p,q)$, then the surface $S = F \cap V_2$ is geometrically incompressible in V_2.*

Proof. We assume, for a contradiction, that S is geometrically compressible. Let D be a compressing disk of S. Because F is geometrically incompressible, D is not a compressing disk of $F \subset L(p,q)$, and ∂D bounds a disk D' in F. Note that D' intersects V_1 since D is a compressing disk of S in V_2. As is well-known, a lens space is irreducible, and hence the sphere $D \cup D'$ bounds a 3-ball, say, B in $L(p,q)$. We move F by isotoping D' along B onto D. The number of meridian disks of $F \cap V_1$ decreases. This is a contradiction. Hence S is geometrically incompressible. $\square$

Let M be a 3-manifold with boundary, and S a 2-manifold properly embedded in M. A disk D in M is called a *∂-compressing disk* if $\alpha = D \cap S$ is a subarc of ∂D, $\beta = D \cap \partial M$ is also a subarc of ∂D, $\partial D = \alpha \cup \beta$, $\alpha \cap \beta = \partial\alpha = \partial\beta$ and α does not cobound a subdisk of S with a subarc of ∂S. S is said to be *∂-compressible* if it has a ∂-compressing disk. Otherwise, S is *∂-incompressible*. (Any closed surface is considered to be ∂-incompressible.)

When S is ∂-compressible, we can obtain a new surface S' by a *∂-compression* as below. We take a tubular neighbourhood $N(D) \cong D \times I$ of D in M so that $N(D) \cap S = \alpha \times I$ and $N(D) \cap \partial M = \beta \times I$. Then we set $S' = (S - \alpha \times I) \cup (D \times \partial I)$.

The next two lemmas are well-known. We omit their proofs.

Lemma 3.3. *, Let V be a solid torus, and S be a (possibly disconnected) 2-manifold properly embedded in V. If S is geometrically incompressible and ∂-incompressible in V, then S is a disjoint union of spheres and disks.*

Lemma 3.4. *Let M be a 3-manifold with boundary, S a (possibly disconnected) 2-manifold properly embedded in M, S' a 2-manifold obtained from S by a ∂-compression. If S is geometrically incompressible in M, then so is S'.*

Proof of Proposition 3.1. Assume that F is isotoped so that it intersects V_1 in the minimum number of meridian disks of V_1. If $F \cap V_1 = \emptyset$, then F is a geometrically incompressible closed surface in V_2, and hence is a sphere, which is a contradiction. So F intersects V_1 in at least one meridian disk. The surface $S = F \cap V_2$ is geometrically incompressible in V_2 by Lemma 3.2. We assume, for a contradiction, that S is ∂-incompressible in V_2. Then S is a disjoint union of spheres and disks by Lemma 3.3, and hence F is a union of spheres. This contradicts that F is connected and not a sphere. Hence S is ∂-compressible.

Claim: $F \cap V_1$ is a single meiridian disk in V_1.

Proof of Claim. We assume, for a contradiction, that $F \cap V_1$ consists of two or more meridian disks in V_1. Let Q be a ∂-compressing disk of S. We move F by an isotopy along Q, to obtain a surface F'. This isotopy causes a ∂-compressing operation on $S = F \cap V_2$ along Q in V_2, and a band sum operation on $F \cap V_1$ along the arc $\beta = Q \cap \partial V_1$ in V_1. There are two cases: the endpoints of β are contained in either distinct two meridian disks of $F \cap V_1$, or a single meridian disk of $F \cap V_1$.

In the first case, the band joins the two meridian disks. They are deformed into a peripheral disk, say, R in V_1, that is, R is isotopic into ∂V_1 with ∂R fixed. Hence we can move F near R so that R is pushed out of V_1. This decreases the number of meridian disks of $F \cap V_1$, which is a contradiction.

We consider the second case, where $\partial\beta$ is contained in a meridian disk D of V_1. The band sum is not essential since boundary circles of other meridian disks prevent β from connecting both sides of the circle ∂D. The endpoints of β separates the circle ∂D into two subarcs. One of them and β together form an inessential circle which bounds a disk, say, E on ∂V_1. $F' \cap E = \partial E$ after the isotopy of F along Q. Since $S' = F' \cap V_2$ is geometrically incompressible by Lemma 3.4, there is a disk $E' \subset S'$ with $\partial E' = \partial E$. E' is a connected component of S', and $\partial E'$ contains precisely one of two copies of β. Hence the disk E' contains exactly one of two copies of Q, and $\mathrm{cl}\,(E' - Q)$ is a disk, which contradicts that Q is a ∂-compressing disk of S. This completes the proof of Claim. $\square$

Now, $F \cap V_1$ is a single meridian disk of V_1. The surface $S = F \cap V_2$ is connected since S is obtained from the closed surface F by removing the meridian disk $F \cap V_1$. S is geometrically incompressible and ∂-compressible in V_2. Let E_1 be a ∂-compressing disk of S. We move F by an isotopy along E_1 to obtain a new surface, say, F_1. Then $S_1 = F_1 \cap V_2$ is obtained from S by a ∂-compression along E_1, and hence is geometrically incompressible by Lemma 3.4. $F_1 \cap V_1$ is obtained from $F \cap V_1$ by a band sum along the arc $\beta_1 = \partial E_1 \cap \partial V_1$.

In the case where both ends of the band are in the same side of the meridian disk $F \cap V_1$, we obtain a contradiction by a similar argument as that in the latter part of the proof of Claim. Hence the ends of the band are in distinct sides of the meridian disk $F \cap V_1$. Then the band sum along β_1 is essential, and the boundary of the surface $F_1 \cap V_1$ is an essential circle on ∂V_1. $F_1 \cap V_1$ and $S_1 = F_1 \cap V_2$ are both connected surfaces since F_1 is connected and $F_1 \cap \partial V_1$ is a single circle. If S_1 is ∂-incompressible, then S_1 is a disjoint union of spheres, meridian disks and peripheral disks by Lemma 3.3. Since $\partial(F \cap V_2)$ is an essential circle, $S_1 = F_1 \cap V_2$ is a meridian disk of V_2. Thus we obtained the desired conclusion in this case.

We consider the case where S_1 is ∂-compressible in V_2 after the isotopy along the disk E_1. Let E_2 be a ∂-compressing disk of S_1. We move F_1 along E_2 to obtain a new surface

F_2. The surface $S_2 = F_2 \cap V_2$ is obtained from S_1 by a ∂-compression along E_2, and $F_2 \cap V_1$ is obtained from $F_1 \cap V_1$ by a band sum along the arc $\beta_2 = \partial E_2 \cap \partial V_1$. If the band sum along β_2 is inessential, then we obtain a contradiction by the same argument as that in the latter part of the proof of Claim. Hence the band sum along β_2 is essential, $F_2 \cap \partial V_2$ is an essential circle in ∂V_2, and S_2 and $F_2 \cap V_1$ are both connected surfaces. S_2 is geometrically incompressible in V_2 by Lemma 3.4. If S_2 is ∂-incompressible, then S_2 is a meridian disk, and we are done. If S_2 is ∂-compressible, then we perform an isotopy along a ∂-compressing disk, say, E_3.

As long as S_{k-1} is ∂-compressible we repeat an isotopy deforming F_{k-1} along a ∂-compressing disk E_k into a surface F_k, and set $S_k = F_k \cap V_2$. This repetition terminates in finite number of times because the Euler characteristic of S_k is larger than that of S_{k-1} by one. For some natural number $m \in \mathbb{N}$, the surface S_m is ∂-incompressible, and hence is a meridian disk in V_2. This completes the proof. □

4. SURGERIES ON CIRCLES AND SLOPES

We show in this section that a band sum operation in Definition 1.8 corresponds to an edge of $\mathbb{D}_2$. From Definition 4.1 through Definition 4.6, H denotes a surface, C an embedded circle in H, and β an embedded arc in H such that $\beta \cap C = \partial\beta$.

Definition 4.1. A *surgery* on C along β is an operation which deforms C to a new circle(s) C' as below. We take a tubular neighbourhood $N(\beta) \cong \beta \times I$ of β in H so that $N(\beta) \cap C = (\partial\beta) \times I$. Then $C' = (C - (\partial\beta) \times I) \cup (\beta \times \partial I)$ is either a cirle or a disjoint union of two circles.

Remark 4.2. Let F be a surface properly embedded in a 3-manifold M. Suppose ∂F is a circle. Then, setting $H = \partial M$ and $C = \partial F$, a surgery operation on C along β coincides with the deformation of ∂F under the band sum operation on F along β.

The next lemma and Corollary 4.5 were pointed out and used in the proof of Theorem 13 in [7] by Rubinstein.

Lemma 4.3. *Suppose that H and C are oriented. Assume that β connects both sides of C. By performing a surgery on C along β, we obtain a single circle, say, C'. For an arbitrary orientation of C', the algebraic intersection number $C' \cdot C$ is equal to ± 2, and hence C' is essential in H.*

Proof. In Definition 4.1 of surgery on C along β, $C - N(\beta)$ consists of two arcs, say, C_1 and C_2. Since β connects both sides of C and since H is orientable, each of the two arcs of $\beta \times \partial I$ connects an endpoint of C_1 and that of C_2. Hence the surgery yields a single circle, say, C'. For an arbitrary orientation of C', the induced orientation of the two subarcs of

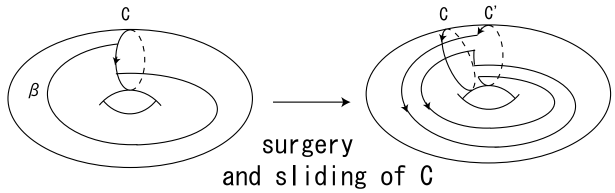

FIGURE 5.

$\beta \times \partial I$ of C' are parallel as a pair of opposite sides of the quadrilateral $\beta \times I$. After the surgery, we slide the original circle C a little so that C intersects $\beta \times I$ in an arc of the form $p \times I$, where p is a point of β. See Figure 5. Then C intersects C' at the endpoints of $p \times I$, and the signs of them coincide. Hence we have $C \cdot C' = \pm 2$, and C' is essential in H. □

Definition 4.4. The two points $\partial\beta$ separate C into two arcs, say, C_1, C_2. If none of the circles $\beta \cup C_1$ and $\beta \cup C_2$ bounds a disk in H, then we say that the surgery on C along the arc β is *essential*.

Corollary 4.5. *If H is a torus, and if a surgery on C along the arc β is essential, then it yields a circle C' such that $C \cdot C' = \pm 2$. Hence C' is essential in H.*

Proof. The arc β connects both sides of C since H is a torus and the surgery is essential. Hence we obtain the desired conclusion by Lemma 4.3. □

Definition 4.6. We assume that a single circle C' is obtained by a surgery on C along the arc β in H. Then we can recover C by performing a *dual surgery* on C' as below. Let p be a point in the interior of the arc β. C can be restored by a surgery on C' along the arc $(p \times I) \subset (\beta \times I)$, where $\beta \times I$ is a square as in Definition 4.1.

Theorem 4.7. *Let T be a torus, and C and C' oriented circles in T. Then (1) and (2) below are equivalent.*

(1) *We can obtain C' by performing a surgery on C along an arc.*
(2) $C' \cdot C = \pm 2$.

Proof. First, we assume (1) to show (2). C' can be obatined by a surgery on C along an arc β. If the surgery is inessential, then it yields two circles one of which bounds a disk in T. This contradicts that C' is a single circle. Hence the surgery is essential, and (2) holds by Corollary 4.5.

Next, we prove (1) under the assumption that (2) holds. It is well-known that we can place C and C' so that C and C' intersect at precisely two points of the same sign since $C \cdot C' = \pm 2$ and T is a torus. The two intersection points separate C into two arcs. Let γ be one of them. Since the signs of the two intersection points coincide, a surgery on C' along the arc γ yields a single circle, say, C''. See Figure 6. Because we have already proven

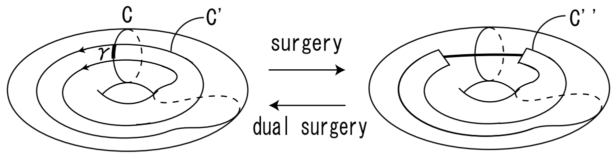

FIGURE 6.

(1) implies (2), $C' \cdot C'' = \pm 2$, and C'' is essential. Then C'' is parallel to C in the torus T since C'' is essential and does not intersect C. C' is obtained by a dual surgery on C'', and hence on C. $\square$

Theorem 4.7 together with Lemma 5.1 in the next section shows Theorem 1.11.

5. $\mathbb{D}_2$

In this section, we prove the graph $\mathbb{D}_2$ is a tree.

Lemma 5.1. *Let p/q and r/s be irreducible fractional numbers. If p is even and $d(p/q, r/s) = 2$, then r is also even.*

Proof. Since $\pm 2 = \pm d(p/q, r/s) = ps - qr$, and since p is even, $qr = ps \mp 2$ is also even. Hence either q or r is even. Because p/q is irreducible, r is even. $\square$

Lemma 5.2. *Let p/q and r/s be irreducible fractional numbers with $d(p/q, r/s) = 2$. If $p/q > 0$ and $r/s < 0$, then $p/q = 1$ and $r/s = -1$.*

Proof. We can assume $p, q, s > 0$ and $r < 0$. Then $ps \geq 1$ and $s(-r) \geq 1$, and $ps + s(-r) \geq 2$. On the other hand, $ps + s(-r) = \pm d(p/q, r/s) = \pm 2$. Hence we have $ps = 1$ and $s(-r) = 1$. Since p, q, r, s are integers, the lemma follows. $\square$

Because neither $1/1$ nor $-1/1$ is a vertex of $\mathbb{D}_2$, the next corollary holds.

Corollary 5.3. *The graph $\mathbb{D}_2$ has no edge connecting the left hand side and the right hand side of the Poincaré disk.*

Notation 5.4. Let $\mathbb{D}_{2+}$ denote the right half of the graph $\mathbb{D}_2$. That is, vertices of $\mathbb{D}_{2+}$ are those of $\mathbb{D}_2$ corresponding to non-negative rational numbers and edges of it are those of $\mathbb{D}_2$ connecting pairs of vertices of $\mathbb{D}_{2+}$. Similarly, we define $\mathbb{D}_{2-}$ as the left half of $\mathbb{D}_2$. Then $\mathbb{D}_2 = \mathbb{D}_{2-} \cup \mathbb{D}_{2+}$, and $\mathbb{D}_{2-} \cap \mathbb{D}_{2+} = \{0/1\}$.

Lemma 5.5. *$\mathbb{D}_2$ is symmetric about the line connecting $0/1$ and $1/0$. Precisely, $d(p/q, p'/q') = d(-p/q, -p'/q')$ for any pair of irreducible fractional numbers p/q and p'/q'.*

Proof. $d(-\frac{p}{q}, -\frac{p'}{q'}) = |\det \begin{pmatrix} -p & -p' \\ q & q' \end{pmatrix}| = |-\det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}| = |\det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}| = d(\frac{p}{q}, \frac{p'}{q'})$ $\square$

Notation 5.6. For a finite sequence of real numbers $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, let $\phi_{\mathbf{a}} : \mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ denote a homeomorphism defined by $\phi_{\mathbf{a}}(x) = [a_n, a_{n-1}, \cdots, a_2, a_1, x]$. If $n = 0$, then $\phi_{\mathcal{A}}(x) = x$.

Remark 5.7. We set $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. For any $r \in \mathbb{R}_+$, the map $x \mapsto x + r$ preserves the order in $\mathbb{R}_+ \cup \{\infty\}$ (i.e., $x + r > y + r$ if $x, y \in \mathbb{R}_+ \cup \{\infty\}$ and $x > y$), while the map $x \mapsto 1/x$ reverses it. Hence, for a finite sequence of non-negative real numbers $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\phi_{\mathbf{a}}$ preserves the order in $\mathbb{R}_+ \cup \{\infty\}$ when n is even, and reverses it when n is odd.

Lemma 5.8. *For any finite sequence of integers $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, there are integers α, β, γ and δ such that for any irreducible fractional number p/q, (1) $\phi_{\mathbf{a}}(p/q) = \dfrac{\alpha p + \beta q}{\gamma p + \delta q}$, (2) $\alpha p+\beta q$ and $\gamma p+\delta q$ are coprime and (3) $\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (-1)^n$. Moreover, if $a_1, a_2, \cdots, a_n$ are non-negative, we can take $\alpha, \beta, \gamma, \delta$ to be non-negative.*

Proof. We show this lemma by induction on n. When $n = 0$, $\phi_{\mathcal{A}}(p/q) = p/q = \dfrac{1 \cdot p + 0 \cdot q}{0 \cdot p + 1 \cdot q}$, and the lemma holds.

We assume that there are integers $\alpha', \beta', \gamma', \delta'$ as above for $(a_1, a_2, \cdots, a_{k-1})$.

When $n = k$, $\phi_{\mathcal{A}}(p/q) = [a_k, a_{k-1}, \cdots, a_2, a_1, p/q]$ $= [a_k, \dfrac{\alpha' p + \beta' q}{\gamma' p + \delta' q}] = a_k + \dfrac{\gamma' p + \delta' q}{\alpha' p + \beta' q} = \dfrac{(a_k\alpha' + \gamma')p + (a_k\beta' + \delta')q}{\alpha' p + \beta' q}$. We set $\alpha = a_k\alpha' + \gamma'$, $\beta = a_k\beta' + \delta'$, $\gamma = \alpha'$ and $\delta = \beta'$.

Then, $\alpha p + \beta q$ and $\gamma p + \delta q$ are coprime since $\alpha p + \beta q = a_k(\gamma p + \delta q) + (\gamma' p + \delta' q)$ and since $\gamma p + \delta q = \alpha' p + \beta' q$ and $\gamma' p + \delta' q$ are coprime by the assumption of induction.

In addition, $\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \det\begin{pmatrix} a_k\alpha' + \gamma' & a_k\beta' + \delta' \\ \alpha' & \beta' \end{pmatrix} = \det\begin{pmatrix} \gamma' & \delta' \\ \alpha' & \beta' \end{pmatrix} =$ $-\det\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = -(-1)^{k-1}$, where we obtain the first equation by subtracting a_k times the second row from the first row, and the last equation by the assumption of induction.

If $a_1, a_2, \cdots, a_k$ are non-negative, then $\alpha = a_k\alpha' + \gamma'$, $\beta = a_k\beta' + \delta'$, $\gamma = \alpha'$ and $\delta = \beta'$ are non-negative since a_k is non-negative and α', β', γ' and δ' are non-negative by the assumption of induction. $\square$

Lemma 5.9. *For any finite sequence of integers $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, the map $\phi_{\mathbf{a}}$ preserves the distance.*

Proof. Let p/q and r/s be irreducible fractional numbers. As in Lemma 5.8, there are integers α, β, γ and δ with $\phi_{\mathbf{a}}(t/u) = (\alpha t+\beta u)/(\gamma t+\delta u)$ for any irreducible fractional number

t/u. Hence $d(\phi_{\mathbf{a}}(p/q), \phi_{\mathbf{a}}(r/s)) = |\det \begin{pmatrix} \alpha p + \beta q & \alpha r + \beta s \\ \gamma p + \delta q & \gamma r + \delta s \end{pmatrix}| = |\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix}| = |(-1)^n \det \begin{pmatrix} p & r \\ q & s \end{pmatrix}| = d(p/q, r/s)$ □

Definition 5.10. For any positive irreducible fractional number p/q with p even and p, q positive, we define the *mother* $M(p/q)$ of p/q as below. Let $p/q = [a_0, a_1, \cdots, a_{n-1}, a_n]$ be the standard continued fraction expansion. Then we set $M(p/q) = [a_0, a_1, \cdots, a_{n-1}, a_n - 2]$. Then $M(p/q)$ is a non-negative rational number which is expressed by an irreducible fractional number with its numerator even, which is shown in Lemma 5.13.

Remark 5.11. $d(p/q, M(p/q)) = 2$ by Lemma 5.9 and $d(a_n, a_n - 2) = 2$. Hence the vertices p/q and $M(p/q)$ are connected by an edge in the graph $\mathbb{D}_2$.

Definition 5.12. For an irreducible fractional number p/q with $p \geq 0$ and $q > 0$, we define *size* of p/q as $\text{size}(p/q) = p + q$. For $\infty = 1/0$, $\text{size}(\infty) = 1 + 0 = 1$.

Lemma 5.13. *For any positive irreducible fractional number p/q with p even and p, q positive, the mother $M(p/q)$ is a non-negative rational number which is expressed by an irreducible fractional number with its numerator even. Moreover,* $\text{size}(M(p/q)) < \text{size}(p/q)$

Proof. As in Definition 5.10, there is a unique continued fraction expansion $p/q = [a_0, a_1, \cdots, a_{n-1}, a_n]$. For the sequence $\mathbf{a} = (a_{n-1}, a_{n-2}, \cdots, a_1, a_0)$, there are non-negative integers $\alpha, \beta, \gamma, \delta$ such that $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (-1)^n$ and $\phi_{\mathbf{a}}(r/s) = \dfrac{\alpha r + \beta s}{\gamma r + \delta s}$ for any irreducible fractional number r/s as in Lemma 5.8.

Then $\dfrac{p}{q} = \phi_{\mathbf{a}}(a_n) = \dfrac{\alpha \cdot a_n + \beta \cdot 1}{\gamma \cdot a_n + \delta \cdot 1}$, and $M(\dfrac{p}{q}) = \phi_{\mathbf{a}}(a_n - 2) = \dfrac{\alpha(a_n - 2) + \beta \cdot 1}{\gamma(a_n - 2) + \delta \cdot 1}$. These expression of fractional numbers are irreducible by Lemma 5.8. Since $p = \alpha a_n + \beta \cdot 1$ is even, (the numerator of $M(p/q)) = \alpha(a_n - 2) + \beta \cdot 1$ is also even.

Moreover, size $(p/q) = a_n(\alpha + \gamma) + \beta + \delta$, and size $(M(p/q)) = (a_n - 2)(\alpha + \gamma) + \beta + \delta$. Hence size $(p/q) - \text{size}(M(p/q)) = 2(\alpha + \gamma) > 0$. Note that either $\alpha > 0$ or $\gamma > 0$ holds because they are non-negative and $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0$. □

Lemma 5.14. *The graph $\mathbb{D}_2$ is connected. In fact, for any irreducible fractional number p/q with p even, $p \geq 0$ and $q > 0$, the sequence*
$p/q, M(p/q), M(M(p/q)), M(M(M(p/q))), \cdots, M^k(p/q), \cdots$
reaches $0/1$. That is, $M^m(p/q) = 0/1$ for some non-negative integer m.

Proof. By Lemma 5.5 and $\mathbb{D}_{2-} \cap \mathbb{D}_{2+} = \{0/1\}$, it is enough to show that $\mathbb{D}_{2+}$ is connected.

If $M^k(p/q) > 0$, then we take $M^{k+1}(p/q)$, which is of smaller size than $M^k(p/q)$ by Lemma 5.13. This repetition terminates at most size$(p/q) = p + q$ times. Hence $\mathbb{D}_{2+}$ is connected. □

Definition 5.15. Let p, q be coprime integers with p even, $p \geq 0$ and $q > 0$. Then an irreducible fractional number r/s is a *child* of p/q if $d(r/s, p/q) = 2$ and r/s is not the mother of p/q.

Definition 5.16. Let p and q be coprime integers with p even, $p \geq 0$ and $q > 0$. We say that the irreducible fractional number p/q is of the kth *generation* if $M^k(p/q) = 0/1$. We set $\mathbb{D}_{2k+}$ to be a subgraph of $\mathbb{D}_{2+}$ such that its vertices are the vertices of $\mathbb{D}_{2+}$ of k or smaller generation and its edges are those of $\mathbb{D}_{2+}$ with endpoints at vertices of k or smaller generations.

Definition 5.17. Let p, q be coprime integers with p even, $p \geq 0$ and $q > 0$. For the irreducible fractional number p/q, we define the territory $T(p/q)$ of p/q as a certain open interval in $\mathbb{R}$ below. Let $p/q = [a_0, a_1, \cdots, a_{n-1}, a_n]$ be the standard continued fraction expansion.

When n is even, $T(p/q) = ([a_0, a_1, \cdots, a_{n-1}, a_n - 1],\ [a_0, a_1, \cdots, a_{n-1}, \infty])$.

When n is odd, $T(p/q) = ([a_0, a_1, \cdots, a_{n-1}, \infty],\ [a_0, a_1, \cdots, a_{n-1}, a_n - 1])$.

Lemma 5.18. *Let p, q be coprime positive integers with p even, and $p/q = [a_0, a_1, \cdots, a_{n-1}, a_n]$ the standard continued fraction expansion. A rational number r/s is a child of p/q if and only if $r/s = [a_0, a_1, \cdots, a_{n-1}, a_n + 2/t]$ for some odd integer t other than -1. p/q is the mother of its child, and hence if p/q is of the gth generation, then its child is of the $(g + 1)$st generation. The territory $T(p/q)$ contains p/q and all the children of p/q, and $M(p/q) \notin T(p/q)$.*

Proof. Let u/w be an irreducible fractional number with $d(a_n, u/w) = 2, u \geq 0$ and $w > 0$. Then $u = a_n w \pm 2$. If w is even, then u is also even, which contradicts that u/w is irreducible. Hence w is odd. Dividing both sides by w, we have $u/w = a_n \pm 2/w$.

Hence, by Lemma 5.9, an irreducible fractional number r/s with $d(r/s, p/q) = 2$ is of the form $r/s = [a_0, a_1, \cdot, a_{n-1}, a_n + 2/t]$ for some odd integer t. This is the mother of p/q when $t = -1$.

When $t \neq -1$, an easy calculation shows that p/q is the mother of r/s. For example, when $t \leq -5$, we have a continued fraction expansion $r/s = [a_0, a_1, \cdots, a_{n-1}, a_n - 1, 1, (-t-3)/2, 2]$. Hence the mother of r/s is $M(r/s) = [a_0, a_1, \cdots, a_{n-1}, a_n - 1, 1, (-t-3)/2, 2-2] = [a_0, a_1, \cdots, a_n] = p/q$.

Remark 5.7 and the sequence of inequalities below imply that $p/q \in T(p/q)$, any child of p/q is in $T(p/q)$ and $M(p/q) \notin T(p/q)$.

$$\infty > a_n + 2/1 > a_n + 2/3 > a_n + 2/5 > \cdots > a_n > \\ \cdots > a_n - 2/5 > a_n - 2/3 > a_n - 1 > a_n - 2 \geq 0$$

□

Lemma 5.19. *Let p, q be coprime positive integers with p even. For any child r/s of p/q, $T(r/s) \subset T(p/q)$ and $p/q \notin T(r,s)$. Let r'/s' be another child of p/q. Then $T(r'/s') \cap T(r/s) = \emptyset$.*

Proof. Let $p/q = [a_0, a_1, \cdots, a_{n-1}, a_n]$ be the standard continued fraction expansion. Then $r/s = [a_0, a_1, \cdots, a_{n-1}, a_n + 2/t]$ for some odd integer t other than -1 by Lemma 5.18.

We show the lemma in the case where n is even. (If n is odd, then the order in $\mathbb{R}$ is reversed by the map $\phi_{\mathbf{a}}$ with $\mathbf{a} = (a_{n-1}, a_{n-2}, \cdots, a_0)$. However, a similar argument as below shows the lemma.)

An easy calculation shows that
$T(r/s) = ([a_0, a_1, \cdots, a_{n-1}, a_n + 2/(t+1)],\ [a_0, a_1, \cdots, a_{n-1}, a_n + 2/(t-1)])$.
(Note that $2/(t-1) = \infty$ when $t = 1$.) For example, when $t \leq -5$, we have a continued fraction expansion $r/s = [a_0, a_1, \cdots, a_{n-1}, a_n - 1, 1, (-t-3)/2, 2]$. Hence $T(r/s) = ([a_0, a_1, \cdots, a_{n-1}, a_n - 1, 1, (-t-3)/2, \infty],\ [a_0, a_1, \cdots, a_{n-1}, a_n - 1, 1, (-t-3)/2, 2-1])$, which coincides with the above open interval.

Since $a_n - 2/2 < a_n - 2/4 < a_n - 2/6 < \cdots < a_n < \cdots < a_n + 2/4 < a_n + 2/2 < \infty$, Remark 5.7 implies that the territory $T(r/s)$ is contained in $([a_0, a_1, \cdots, a_{n-1}, a_n - 1],\ p/q)$ when t is negative, and in $(p/q,\ [a_0, a_1, \cdots, a_{n-1}, \infty])$ when t is positive, and the lemma holds. □

Proof of Theorem 1.12. We show that $\mathbb{D}_2$ is a tree. By Lemma 5.14, $\mathbb{D}_2$ is connected. Hence we have only to show that $\mathbb{D}_2$ contains no cycle. Since $\mathbb{D}_2 = \mathbb{D}_{2+} \cup \mathbb{D}_{2-}$ and $\mathbb{D}_{2+} \cap \mathbb{D}_{2-} = \{0/1\}$, and since $\mathbb{D}_2$ is symmetric about the line connecting $0/1$ and $1/0$, it is sufficient to show that $\mathbb{D}_{2+}$ contains no cycle.

Lemmas 5.18 and 5.19 together imply that $\mathbb{D}_{2(m+1)+}$ retracts to $\mathbb{D}_{2m+}$ for any positive integer m. Hence $\mathbb{D}_{2k+}$ does not contain a cycle.

$\mathbb{D}_{2+} = \cup_{i=1}^{\infty} \mathbb{D}_{2i+}$ by Lemma 5.14. If $\mathbb{D}_{2+}$ contained a cycle, then also $\mathbb{D}_{2k+}$ would contain a cycle for some positive integer k. □

6. The proof of the main theorem

Lemma 6.1. *Let F be a surface in standard form as in Definition 1.8, and b the number of the band sum operations. Then, the Euler characteristic of F is calculated by $\chi(F) = 2 - b$, and hence the crosscap number of F is $Cr(F) = b$.*

Proof. We can regard the meridian disk in V_1 as a 0-handle, each band as a 1-handle, the meridian disk in V_2 as a 2-handle. □

Lemma 6.2. *Let $V \cong D^2 \times I$ be a solid torus, C a circle of (p,q)-slope on ∂V, and r/s an irreducible fractional number with $d(r/s, p/q) = 2$. Then there is a unique arc β up to*

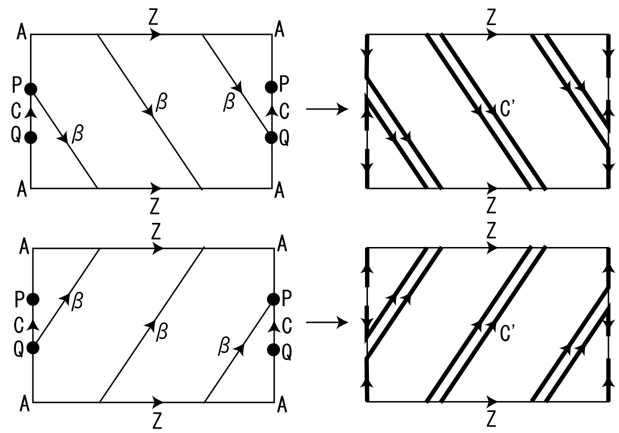

FIGURE 7.

ambient isotopy of V fixing C as a set such that a surgery on C along β yields a circle of (r,s)-slope.

Proof. The existence follows from Theorem 4.7. We show the uniqueness. We give C an arbitrary orientation. There is an oriented circle Z in ∂V such that Z intersects C transversely in a single point, say, A, and $Z \cdot C = +1$. We fix Z. Let β be an arc embedded in ∂V such that $\beta \cap C = \partial\beta$. Let C' be the circle obtained from C by a surgery along β. We orient C' so that $C' \cdot C = +2$, which induces an orientation of β. It is sufficient to show that distinct ambient isotopy classes of β give distinct intersection numbers $C' \cdot Z$. Let P and Q denote endpoints of β so that A, Q, P appear in this order on the oriented circle C. Assume that β intersects Z transversely in the nimimum number of points up to ambient isotopy of V fixing C as a set. Let k be the minimum number. Then $C' \cdot Z = +(2k+1)$ or $-(2k+1)$ according as β starts at P or Q. See Figure 7, where the torus ∂V cut along $C \cup Z$ is described. □

Note that the argument below shows the uniqueness of the isotopy class of geometrically incompressible closed surface in $L(p,q)$ with p even.

Proof of Theorem 1.13. By Proposition 3.1, F is isotopic to a surface in standard form. The transition of slopes by band sum operations is along an edge-path, say, ρ in $\mathbb{D}_2$ as mentioned in right after Theorem 1.11. ρ starts at $0/1$ and ends at p/q.

Since $\mathbb{D}_2$ is a tree by Theorem 1.12, there is a unique minimal edge-path $\gamma(p,q)$ connecting $0/1$ and p/q such that $\gamma(p,q)$ passes each edge of $\mathbb{D}_2$ at most once. The union of the edges of ρ contains $\gamma(p,q)$.

Suppose, for a contradiction, that $\rho \neq \gamma(p,q)$. Then ρ passes the same edge of $\mathbb{D}_2$ twice consecutively. Hence corresponding two band sum operations cause mutually dual surgeries on boundary circles on ∂V_1 by Lemma 6.2. These two bands together form an

annulus whose core circle bounds a compressing disk Q of the surface F_i in V_1 such that ∂Q is non-separating in F_i. This contradicts that F is geometrically incompressible.

Thus $\rho = \gamma(p, q)$. (This and Lemma 6.2 together imply that a surface in standard form is unique up to isotopy.) Then the theorem follows from Lemmas 6.1 and 5.14. □

Proof of Theorem 1.10. The number of edges in the edge-path $\gamma(p, q)$ in Theorem 1.13 is equal to the number of the band sum operations and to the minimum crosscap number.

Let $p/q = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_1, \alpha_0]$ be the standard continued fraction expansion. If $\alpha'_0 = \alpha_0 = 2\beta_0+1$ for some $\beta_0 \in \mathbb{N}$, then $M^{\beta_0}(p/q) = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_1, 1] = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_1+1]$. We set $\alpha'_1 = \alpha_1 + 1$, which is the last term. If $\alpha'_0 = \alpha_0 = 2\beta_0$ for some $\beta_0 \in \mathbb{N}$, then $M^{\beta_0}(p/q) = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_2, \alpha_1, 0] = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_2, \alpha_1 + 1/0] = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_2, \infty]$. We set $\alpha'_1 = \infty$.

When $\alpha'_1 = \infty$, $[\alpha_n, \alpha_{n-1}, \cdots, \alpha_2, \infty] = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_2 + 1/\infty] = [\alpha_n, \alpha_{n-1}, \cdots, \alpha_2]$. We set $\beta_1 = 0$ and $\alpha'_2 = \alpha_2$. When $\alpha'_1 = 2\beta_1 + 1$ ($\beta_1 \in \mathbb{N}$), we set $\alpha'_2 = \alpha_2 + 1$. When $\alpha'_1 = 2\beta_1$ ($\beta_1 \in \mathbb{N}$), we set $\alpha'_2 = \infty$.

We repeat operations of going up to the mother as above. Let $[\alpha_n, \alpha_{n-1}, \cdots, \alpha_i, \alpha'_{i-1}]$ be the continued fraction expansion of length $n - (i - 2)$ which we first reach. Then we set $\alpha'_i = \alpha_i$ (when $\alpha'_{i-1} = \infty$), $\alpha_i + 1$ (when $\alpha'_{i-1} = 2\beta_{i-1} + 1$ for some $\beta_{i-1} \in \mathbb{N} \cup \{0\}$) and ∞ (when $\alpha'_{i-1} = 2\beta_{i-1}$ for some $\beta_{i-1} \in \mathbb{N}$) in a similar way as above.

When we first reach the continued fraction of length one, say, $[\alpha'_n]$, the non-negative integer α'_n is even because it is a vertex of $\mathbb{D}_2$. Hence we set $\alpha'_n = 2\beta_n$. Then $M^{\beta_n}(\alpha'_n) = 0/1$. Thus the number of operations of going up to the mother from p/q to $0/1$ is $\sum_{i=0}^{n} \beta_i$, which is equal to the crosscap number. □

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